

Information and noise in photon entanglement

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Abstract

By using finite resolution measurements it is possible to simultaneously obtain noisy information on two non-commuting polarization components of a single photon. This method can be applied to a pair of entangled photons with polarization statistics that violate Bell's inequalities. The theoretically predicted results show that the non-classical nature of entanglement arises from negative joint probabilities for the non-commuting polarization components. These negative probabilities allow a "disentanglement" of the statistics, providing new insights into the non-classical properties of quantum information.

Keywords: Bell's inequalities, photon statistics, entanglement

1 Introduction

One of the most impressive demonstrations of the specifically non-classical features of quantum mechanics is the violation of Bell's inequalities by entangled photon pairs [1, 2, 3, 4]. The violation of Bell's inequalities shows that it is impossible to explain the statistical predictions of quantum theory by assigning a complete set of polarization components to each photon before the measurement. This implies that the measurement results for a specific polarization direction should not be interpreted as a general property of the photon. In particular, photons cannot be classified as either x or y polarized, even though only these two outcomes are observed in precise polarization measurements along these axes.

In the following, the non-classical correlations of entangled photons are analyzed by applying finite resolution measurements [5, 6, 7]. A measurement setup for the simultaneous measurement of non-commuting polarization components is presented in section 2. In section 3, this measurement concept is applied to entangled photon pairs. It is shown how information on all four polarization components responsible for the viola-

tion of Bell's inequalities can be obtained from a single measurement setup. The analysis of the measurement statistics shows that the violation of Bell's inequalities arises from negative joint probabilities similar to the ones obtained in the single photon measurement setup. The statistics derived from the finite resolution measurement thus allows an identification of the local non-classical properties responsible for the violation of Bell's inequalities.

2 Single photon polarization

The polarization of light can be characterized by the Stokes parameters, defined as the intensity difference between orthogonally polarized modes. A complete description of polarization requires three Stokes parameters. All Stokes parameters can then be written as components of this three dimensional vector. In terms of the annihilation operators for right and left circular polarization, \hat{a}_R and \hat{a}_L , the Stokes parameters read

$$\begin{aligned}\hat{s}_1 &= \hat{a}_R^\dagger \hat{a}_L + \hat{a}_L^\dagger \hat{a}_R \\ \hat{s}_2 &= -i \left(\hat{a}_R^\dagger \hat{a}_L - \hat{a}_L^\dagger \hat{a}_R \right) \\ \hat{s}_3 &= \hat{a}_R^\dagger \hat{a}_R - \hat{a}_L^\dagger \hat{a}_L.\end{aligned}\quad (1)$$

For a single photon, these operators have eigenvalues of ± 1 , as observed in measurements using polarization filters. However, a polarization filter is only sensitive to one component of the Stokes vector at a time, while completely randomizing the information potentially carried by the other two components. This limitation can be overcome by applying finite resolution measurements to obtain information on one polarization component while limiting the noise introduced in the other components. It is then possible to study correlations between the non-commuting polarization components of a single photon.

Figure 1 illustrates the experimental setup for a finite resolution measurement of two orthogonal components of the Stokes vector, \hat{s}_1 and \hat{s}_2 . A beam displacer is

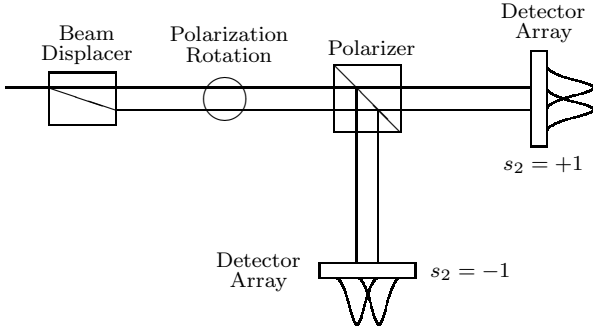


Figure 1: Finite resolution measurement of non-commuting polarization components for a single photon.

used to couple the transversal position of the photon with the polarization component \hat{s}_1 . The resolution of this measurement is given by the ratio of the displacement and the width of the beam. After the measurement of \hat{s}_1 , the polarization component \hat{s}_2 is measured by a $\pi/4$ rotation of the polarization axes and a polarizer. However, the resolution of the \hat{s}_2 measurement is limited by the polarization noise induced in the beam displacer. The detector arrays record the continuous measurement values s_{1m} obtained in the measurement of \hat{s}_1 for the two final measurement values of \hat{s}_2 .

The finite resolution measurement of \hat{s}_1 is described by the measurement operator [5],

$$\hat{P}_{\delta s}(s_{1m}) = (2\pi\delta s^2)^{-\frac{1}{4}} \exp\left(-\frac{(\hat{s}_1 - s_{1m})^2}{4\delta s^2}\right). \quad (2)$$

The probability of a measurement of s_{1m} followed by a measurement of s_2 for an input state $|\psi_{\text{in}}\rangle$ is then given by

$$P(s_{1m}; s_2 = \pm 1) = |\langle s_2 = \pm 1 | \hat{P}_{\delta s}(s_{1m}) | \psi_{\text{in}} \rangle|^2. \quad (3)$$

If the input state is in the $+1$ eigenstate of \hat{s}_2 , the measurement statistics are

$$\begin{aligned} P(s_{1m}; s_2 = +1) &= \frac{1}{\sqrt{2\pi\delta s^2}} \exp\left(-\frac{s_{1m}^2 + 1}{2\delta s^2}\right) \cosh^2\left(\frac{s_{1m}}{2\delta s^2}\right) \\ P(s_{1m}; s_2 = -1) &= \frac{1}{\sqrt{2\pi\delta s^2}} \exp\left(-\frac{s_{1m}^2 + 1}{2\delta s^2}\right) \sinh^2\left(\frac{s_{1m}}{2\delta s^2}\right), \end{aligned} \quad (4)$$

as shown in figure 2 for a resolution of $\delta s = 0.6$. The results show that the intuitive assumption that s_1 should be statistically independent of s_2 is wrong even for an eigenstate of \hat{s}_2 . Instead, the high values of s_{1m} are clearly correlated with “quantum jumps” to $\hat{s}_2 = -1$.

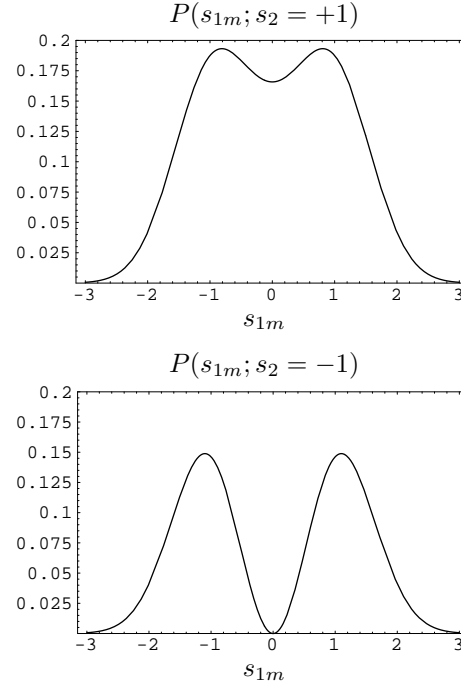


Figure 2: Single photon polarization statistics for photons polarized along the $x = y$ direction ($s_2 = +1$ eigenstates). The resolution is $\delta s = 0.6$ for the initial s_1 measurement in the beam displacer.

As discussed in [7], this implies non-vanishing probability contributions from $s_2 = -1$ in the statistics of the $s_2 = +1$ eigenstate.

s_2	s_1		
	-1	0	$+1$
$+1$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
-1	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$

Table 1: Joint probabilities obtained from the finite resolution measurement setup shown in figure 1 for the $s_2 = +1$ eigenstate.

If the measurement statistics is interpreted in terms of Gaussian contributions with a variance of δs^2 centered around the actual values of s_1 , it appears that, in addition to the quantized eigenvalue results of $s_1 = \pm 1$, results of $s_1 = 0$ must also be taken into account. Moreover, the total probability for $s_1 = 0$ remains zero because the negative joint probability of $-1/2$ for $s_1 = 0$ and $s_2 = -1$ cancels the positive joint probability of $+1/2$ for $s_1 = 0$ and $s_2 = +1$. This negative joint

probability also explains the coexistence of correlations between s_1 and s_2 with a total probability of zero for $s_2 = -1$. The full set of joint probabilities obtained for $\delta s \rightarrow \infty$ is shown in table 1.

Finite resolution measurements thus reveal that negative joint probabilities are an integral part of local quantum statistics. Since this property of quantum statistics contradicts the assumptions made about elements of reality in the formulation of Bell's inequalities [1], it is possible to explain the violation of Bell's inequalities by applying the same analysis to the polarization statistics of entangled photon pairs.

3 Polarization statistics of entangled photons

Figure 3 shows the setup for a coincidence measurement for entangled photons. The two branches a and b are set up as illustrated in figure 1. A maximal violation of Bell's inequalities is obtained for an input state of

$$|\psi_{a,b}\rangle = \frac{1}{\sqrt{2}} \left(|R; L\rangle + \exp\left(-i\frac{\pi}{4}\right) |L; R\rangle \right), \quad (5)$$

where the letters R and L denote eigenstates of right and left polarization for photon a and photon b , respectively. This input state is an eigenstate of the correlation

$$\begin{aligned} \hat{K} = & \hat{s}_1(a)\hat{s}_1(b) + \hat{s}_2(a)\hat{s}_1(b) \\ & - \hat{s}_1(a)\hat{s}_2(b) + \hat{s}_2(a)\hat{s}_2(b) \end{aligned} \quad (6)$$

with an eigenvalue of $K = 2\sqrt{2}$. The maximal value obtained by assigning eigenvalues of ± 1 to the operators $\hat{s}_i(a/b)$ in equation (6) is $K = 2$. Therefore, $|\psi_{a,b}\rangle$ maximally violates the Bell's inequality $K \leq 2$.

Figure 4 shows the measurement statistics obtained for a resolution of $\delta s = 2$. At this resolution, the quantum noise introduced in the measurement of \hat{s}_1 is still very low, so that the original properties of \hat{s}_2 are preserved. Therefore, the statistics clearly reveal the non-classical features of correlations between \hat{s}_1 and \hat{s}_2 . In particular, the peaks of the results obtained for $s_2(a) = -s_2(b)$ are at values of $s_{1m} = \pm\sqrt{2}$, far beyond the eigenvalue limits of ± 1 . Moreover, the peaks are actually sharper than the resolution of $\delta s = 2$ would allow in a classical context.

As in the case of a single photon, the statistics may be interpreted as a sum of Gaussian contributions with a variance of δs^2 centered around the actual values of s_1 . The non-classical features then arise from the negative joint probabilities at $s_1(a) = 0$ and/or $s_1(b) = 0$. The sharpness and the shift of the peaks

at $s_2(a) = -s_2(b)$ are explained by the negative probability at $s_1(a) = s_1(b) = 0$. Table 2 shows the full set of joint probabilities obtained from the measurement results for $\delta s \rightarrow \infty$.

As shown in table 3, each of the 36 measurement results in table 2 corresponds to a well defined value of K . In accordance with the probability maxima in figure 4, the values of $K = +2$ are found at $s_1(a) = +1$ or $s_1(b) = -1$ for $s_2(a) = s_2(b) = -1$, at $s_1(a) = -1$ or $s_1(b) = +1$ for $s_2(a) = s_2(b) = +1$, at $s_1(a) = -1$ and $s_1(b) = -1$ for $s_2(a) = -1$ and $s_2(b) = +1$, and at $s_1(a) = +1$ and $s_1(b) = +1$ for $s_2(a) = +1$ and $s_2(b) = -1$. The broadness of the peaks in the measurement statistics observed for $s_2(a) = s_2(b)$ in figure 4 is explained by the positive probability contribution for $K = +1$ at $s_1(a) = s_1(b) = 0$. The steep slopes of the peaks for $s_2(a) = -s_2(b)$ is likewise explained by the negative probability contribution for $K = -1$ at $s_1(a) = s_1(b) = 0$. The regions of low probability in figure 4 are explained by the near cancellation of negative and positive probabilities for values of $K = -2$ at $s_1(a) = +1$ or $s_1(b) = +1$ for $s_2(a) = -1$ and $s_2(b) = +1$, at $s_1(a) = -1$ or $s_1(b) = -1$ for $s_2(a) = +1$ and $s_2(b) = -1$, at $s_1(a) = -1$ and $s_1(b) = +1$ for $s_2(a) = s_2(b) = -1$, and at $s_1(a) = +1$ and $s_1(b) = -1$ for $s_2(a) = s_2(b) = +1$. The total probability distribution of K values then reads

$$\begin{aligned} P(K = 2) &= 103.1\% & P(K = -2) &= -3.1\% \\ P(K = 1) &= 35.4\% & P(K = -1) &= -35.4\% \\ P(K = 0) &= 0\%. \end{aligned} \quad (7)$$

The violation of Bell's inequalities is therefore the result of negative joint probabilities for the non-commuting polarization components of the entangled photon pair.

4 Elements of reality and negative probabilities

The formulation of Bell's inequalities is based on the assumption that the operator variables can be represented by their eigenvalues. This assumption reflects the definition of elements of reality given in the famous paper by Einstein, Podolsky and Rosen (EPR) [8]: "*If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.*" However, this argument breaks down if the statistics include negative probabilities. If one value of a physical quantity has a probability equal unity, it is still possible that another value of the same property

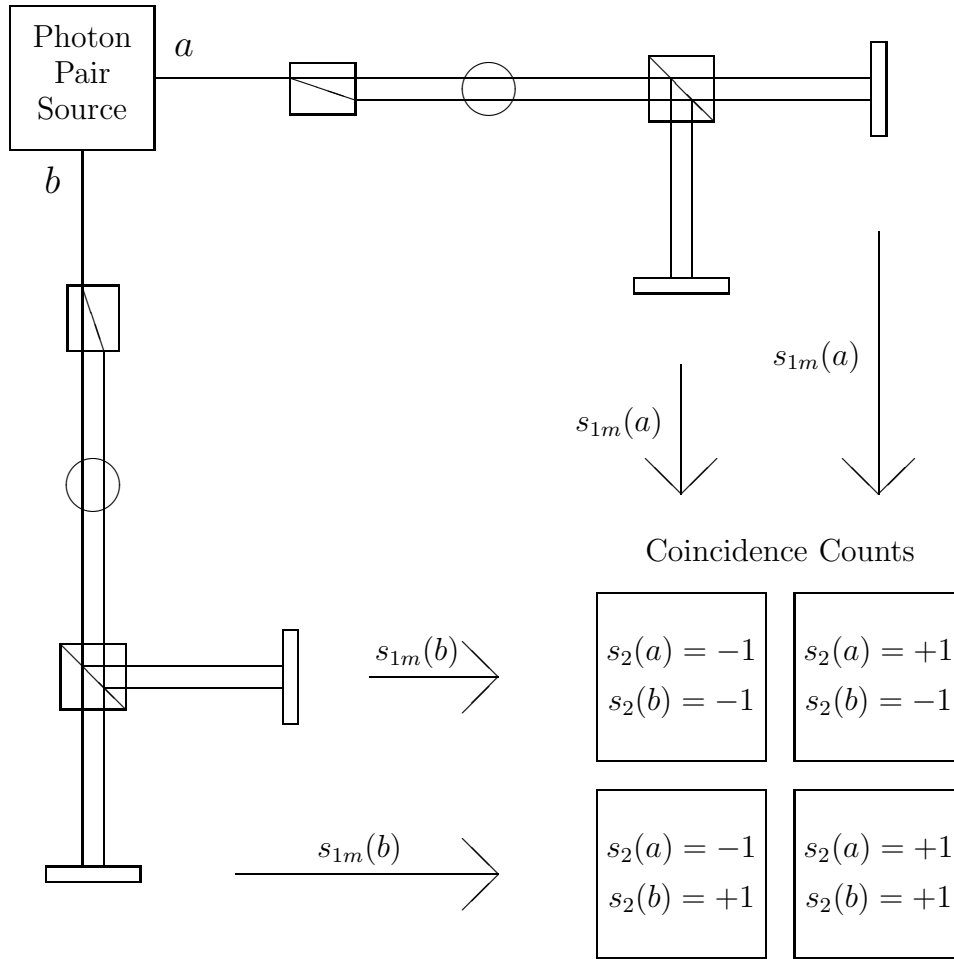


Figure 3: Schematic setup of the coincidence measurement for entangled photons.

has positive and negative joint probabilities. In particular, the discussion of single photon polarization in section 2 revealed the presence of contributions from $s_1 = 0$, even though no eigenvalue of \hat{s}_1 corresponds to this result. While it is possible to predict with certainty that no precise measurement of \hat{s}_1 can produce this result, this certainty does not apply to the result of finite resolution measurements. While the total probability for $s_1 = 0$ is always zero, the joint probabilities shown in tables 1 and 2 are not. Negative probabilities thus introduce a measurement dependent ambiguity into the selection of elements of reality that contradicts the assumptions of Bell's inequalities.

Note that negative probabilities cause no conceptual problems as long as the uncertainty principle applies to all measurements. Indeed, the uncertainty principle can be interpreted as a consequence of negative joint probabilities since it must be impossible to isolate an event associated with a negative probability.

Uncertainty guarantees that negative probabilities are always “covered up” by quantum noise in the measurement process. Effectively, actual measurement results can only be associated with a region of phase space sufficiently large to include more positive than negative probability contributions.

5 Conclusions

Finite resolution measurements of single photon polarization allow simultaneous measurements of non-commuting Stokes parameter components. By applying this type of measurement to entangled photon pairs, details of the violation of Bell's inequalities can be obtained in a single measurement setup. It is possible to represent the statistics of the photon pair polarization in a table of 36 joint probabilities for the non-commuting polarization components. Non-classical features arise from the negative probabilities

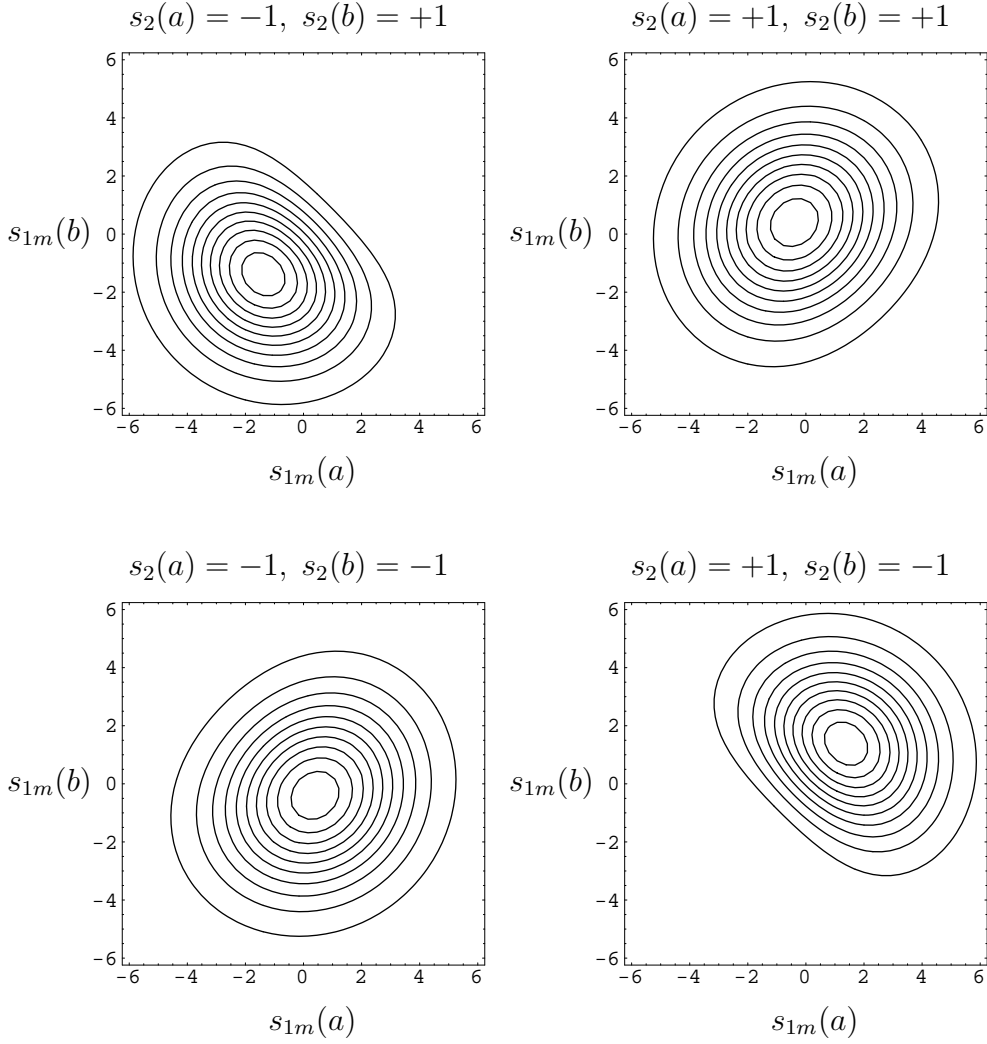


Figure 4: Measurement statistics of the coincidence counts for a resolution of $\delta s = 2$ in the \hat{s}_1 measurements.

at values of $s_1 = 0$. These features not only explain the violation of Bell's inequalities, but also establish a connection between entanglement and the non-classical properties of individual quantum systems.

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$(s_1(b), s_2(b))$	$(s_1(a), s_2(a))$					
	$(-1, -1)$	$(0, -1)$	$(1, -1)$	$(-1, 1)$	$(0, 1)$	$(1, 1)$
$(1, 1)$	$\frac{\sqrt{2}-1}{16\sqrt{2}}$	$-\frac{1}{8\sqrt{2}}$	$\frac{\sqrt{2}+1}{16\sqrt{2}}$	$\frac{\sqrt{2}-1}{16\sqrt{2}}$	$\frac{1}{8\sqrt{2}}$	$\frac{\sqrt{2}+1}{16\sqrt{2}}$
$(0, 1)$	$\frac{1}{8\sqrt{2}}$	$-\frac{1}{4\sqrt{2}}$	$-\frac{1}{8\sqrt{2}}$	$\frac{1}{8\sqrt{2}}$	$\frac{1}{4\sqrt{2}}$	$-\frac{1}{8\sqrt{2}}$
$(-1, 1)$	$\frac{\sqrt{2}+1}{16\sqrt{2}}$	$\frac{1}{8\sqrt{2}}$	$\frac{\sqrt{2}-1}{16\sqrt{2}}$	$\frac{\sqrt{2}+1}{16\sqrt{2}}$	$-\frac{1}{8\sqrt{2}}$	$\frac{\sqrt{2}-1}{16\sqrt{2}}$
$(1, -1)$	$\frac{\sqrt{2}-1}{16\sqrt{2}}$	$-\frac{1}{8\sqrt{2}}$	$\frac{\sqrt{2}+1}{16\sqrt{2}}$	$\frac{\sqrt{2}-1}{16\sqrt{2}}$	$\frac{1}{8\sqrt{2}}$	$\frac{\sqrt{2}+1}{16\sqrt{2}}$
$(0, -1)$	$-\frac{1}{8\sqrt{2}}$	$\frac{1}{4\sqrt{2}}$	$\frac{1}{8\sqrt{2}}$	$-\frac{1}{8\sqrt{2}}$	$-\frac{1}{4\sqrt{2}}$	$\frac{1}{8\sqrt{2}}$
$(-1, -1)$	$\frac{\sqrt{2}+1}{16\sqrt{2}}$	$\frac{1}{8\sqrt{2}}$	$\frac{\sqrt{2}-1}{16\sqrt{2}}$	$\frac{\sqrt{2}+1}{16\sqrt{2}}$	$-\frac{1}{8\sqrt{2}}$	$\frac{\sqrt{2}-1}{16\sqrt{2}}$

Table 2: Joint probabilities for the non-commuting spin components. Note that the total probability for s_1 values of zero is always zero.

$(s_1(b), s_2(b))$	$(s_1(a), s_2(a))$					
	$(-1, -1)$	$(0, -1)$	$(1, -1)$	$(-1, 1)$	$(0, 1)$	$(1, 1)$
$(1, 1)$	-2	-2	-2	+2	+2	+2
$(0, 1)$	0	-1	-2	+2	+1	0
$(-1, 1)$	+2	0	-2	+2	0	-2
$(1, -1)$	-2	0	+2	-2	0	+2
$(0, -1)$	0	+1	+2	-2	-1	0
$(-1, -1)$	+2	+2	+2	-2	-2	-2

Table 3: Values of the correlation K for different joint values of the polarization components.